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1999 J. Phys. A: Math. Gen. 32 3397

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Scheming in dimensional regularization

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Received 9 October 1998, in final form 25 February 1999

Abstract. We consider the most general loop integral that appears in non-relativistic effective field theories with no light particles. The divergences of this integral are in correspondence with simple poles in the space of complex space–time dimensions. Integrals related to the original integral by subtraction of one or more poles in dimensions other than $D = 4$ lead to non-minimal subtraction schemes. Subtraction of all poles in correspondence with ultraviolet divergences of the loop integral leads naturally to a regularization scheme which is precisely equivalent to cut-off regularization. We therefore recover cut-off regularization from dimensional regularization with a non-minimal subtraction scheme. We then discuss the power counting for non-relativistic effective field theories which arises in these alternative schemes.

1. Divergent integrals in non-relativistic effective field theories

Effective field theories of non-relativistic scattering have been a subject of much interest recently. These field theories have applications in many branches of physics. They are useful in analysing a system with widely separated energy scales. For instance, to describe two-body scattering at momenta considerably lower than the mass of any exchanged particle an effective field theory can be written in which all of the exchanged particles are ‘integrated out’ of the original field theory Lagrangian. In the low-energy effective theory the effects of these ‘heavy’ degrees of freedom are represented by an expansion of the interaction Lagrangian as a sequence of local operators of increasing dimension. This low-energy effective theory can then be used to calculate non-relativistic scattering in the system of interest.

In pursuing such a calculation one encounters divergences. These divergences must be regulated and renormalized before physical quantities can be calculated. Here we consider the most general loop integral appearing in a low-energy effective field theory (EFT) description of non-relativistic scattering. We regularize this integral using the standard technique of dimensional regularization (DR). Within the framework of DR, we identify the poles as a function of the number of space–time dimensions that correspond to the divergences of this integral. This allows us to clarify the relation between DR-based schemes and cut-off regularization. We also demonstrate explicitly how the power counting for the coefficients appearing in the non-relativistic effective theory Lagrangian is affected by the choice of regulator.

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Consider a non-relativistic effective field theory for the interaction of two identical heavy particles of mass M . If no exchanged degrees of freedom appear explicitly in the Lagrangian then we can immediately write:

$$\mathcal{L} = \psi^\dagger i \partial_t \psi + \psi^\dagger \frac{\nabla^2}{2M} \psi - \frac{1}{2} C_0 (\psi^\dagger \psi)^2 - \frac{1}{2} C_2 (\psi^\dagger \nabla^2 \psi) (\psi^\dagger \psi) + \text{h.c.} + \dots \quad (1)$$

where the dots refer to other invariants with two or more derivatives. Of course, without a power-counting scheme this Lagrangian is useless, since there are infinitely many interaction terms consistent with assumed symmetries. A power-counting scheme must be established which allows us to keep only a finite number of operators at a given order in the gradient expansion. Unless this is done the effective field theory will have no predictive power, as an infinite number of coefficients will enter the calculation.

We can formally write the all-orders solution of the low-energy effective theory by constructing the non-relativistic potential

$$V(\hat{p}) = \sum_{n=0}^{\infty} C_{2n} \hat{p}^{2n} \quad (2)$$

directly from the Lagrangian. Here \hat{p} is to be understood as an operator, which may denote momentum or energy dependence. More explicitly, this potential is

$$V(\mathbf{k}', \mathbf{k}; E) = \sum_{n=0}^{\infty} \sum_{i=1}^n \sum_j C_{2n}^{(i,j)} p^{2(n-i)} \mathcal{O}_j^i(\mathbf{k}', \mathbf{k}) \quad (3)$$

where $p^2 = ME$ and the index j enumerates members of the set of all operators of dimension $2i$ which are consistent with hermiticity and rotational invariance. In the case of s-wave scattering, the operators that contribute are

$$\mathcal{O}_j^i = k^{2j} k^{2(i-j)} + k^{2(i-j)} k^{2j} \quad (4)$$

although, as we shall see, these details of the construction of the potential are not necessary for our discussions here. Note that the potential V contains only terms that are analytic in the momenta k, k' and p . This is because in writing the effective Lagrangian (1) we have made local expansions of any non-analytic structures which appear in the ‘full’ theory. Therefore, the only non-analytic effects in this low-energy effective theory come from solving the Schrödinger equation. The effective theory is therefore only valid for energies well below the masses or production thresholds of any exchanged particles.

The potential V is then iterated via the Lippmann–Schwinger equation

$$T(k', k; E) = V(k', k; E) + M \int \frac{d^3 q}{(2\pi)^3} V(k', q; E) \frac{1}{EM - q^2 + i\eta} T(q, k; E) \quad (5)$$

to give the on-shell scattering amplitude $T(p, p; E)$, with $p = \sqrt{ME}$. All observables in the $\psi\psi$ system can be obtained from this amplitude. In the language of Feynman diagrams, the Lippmann–Schwinger equation generates the sum of all graphs, as illustrated in figure 1.

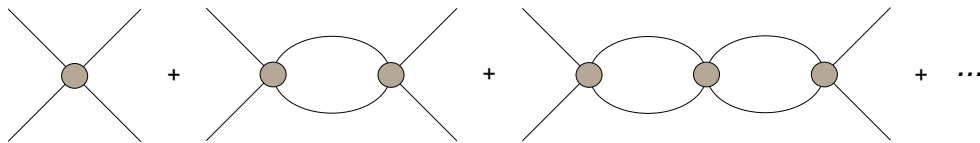


Figure 1. The diagrammatic solution of the Lippmann–Schwinger equation with the effective potential represented by the shaded area.

It immediately follows from equation (5) that divergent integrals will accompany every loop computed in the effective field theory. This means that the C_{2n} coefficients must be renormalized, and therefore are renormalization scheme dependent. Since the scaling behaviour of the C_{2n} coefficients determines which operators are big and which are small in the effective theory, this means that the power counting scheme will look different in different regularization schemes.

In what follows we will consider the most general divergent integral that appears in the effective theory. In section 2 we will explore the many possible definitions of the divergent integral in dimensional regularization, and then in section 3 discuss the power counting for the coefficients C_{2n} that arise from these definitions.

The most general form of the divergent integral which appears in the effective theory is

$$I_n(\mu) = \left(\frac{\mu}{2}\right)^{4-D} M \int \frac{d^{(D-1)}q}{(2\pi)^{D-1}} \frac{q^{2n}}{ME^+ - q^2} \quad (6)$$

where D is the number of space–time dimensions, μ is a renormalization scale, and $E^+ = E + i\eta$, with η a positive infinitesimal, is the energy. Intuition about this integral may be gained by evaluating with a sharp cut-off β . In $D = 4$ this gives

$$I_n(\beta) = M \int \frac{d^3q}{(2\pi)^3} \frac{q^{2n}}{ME^+ - q^2} = \frac{M}{2\pi^2} \int_0^\beta dq \frac{q^{2n+2}}{ME^+ - q^2}. \quad (7)$$

The imaginary part is, of course, regularization scheme invariant. Meanwhile, the real, or principal value, part is given by

$$\text{Re } I_n(\beta) = -\frac{M}{2\pi^2} \left(p^{2n}\beta + \frac{\beta^3}{3} p^{2n-2} + \dots + \frac{\beta^{2n+1}}{2n+1} \right) + p^{2n} \frac{Mp}{4\pi^2} \log \left(\frac{\beta+p}{\beta-p} \right) \quad (8)$$

provided that $\beta^2 > p^2$. It is clear that the integral of interest is ultraviolet power-law divergent at $D = 4$. The maximum power of the cut-off which appears is just the superficial degree of divergence of the graph, $2n + 1$.

Using the usual techniques employed in calculating dimensionally regularized integrals in an arbitrary number of dimensions (see, for instance, appendix B of Ramond [1]), the integral (6) may be re-expressed as,

$$I_n(\mu) = -\left(\frac{\mu}{2}\right)^{4-D} \frac{M}{(4\pi)^{(D-1)/2}} \frac{\Gamma(\frac{2n+D-1}{2})}{\Gamma(\frac{D-1}{2})} (-ME)^{(2n+D-3)/2} \Gamma\left(\frac{3-D-2n}{2}\right) \quad (9)$$

provided that $E < 0$. This equality is valid in a region of the complex D -plane defined by

$$1 - 2n < D < 3 - 2n \quad \text{and} \quad D > 1. \quad (10)$$

The first inequality is easy to understand from equation (6) using simple power counting arguments. The condition $D > 1 - 2n$ ensures that integral is free of infrared divergences and the condition $D < 3 - 2n$ ensures that the integral is free of ultraviolet divergences. According to equation (9), in the complex D -plane (see figure 2) these divergences manifest themselves as singularities of the gamma functions in the corresponding regions. The condition $D > 1$ follows from requiring that the space–time measure be infrared safe.

The two conditions of equation (10) are incompatible for $n > 0$. Thus, in fact, to properly calculate the integral (6) we must do the angular integration in $D = 4$, and then consider the result of varying the number of dimensions in which the ‘radial’ integration is performed. This technique leads to the result:

$$I_n(\mu) = -\left(\frac{\mu}{2}\right)^{4-D} \frac{M}{(4\pi)^{3/2}} \frac{\Gamma(\frac{2n+D-1}{2})}{\Gamma(\frac{3}{2})} (-ME)^{(2n+D-3)/2} \Gamma\left(\frac{3-D-2n}{2}\right) \quad (11)$$

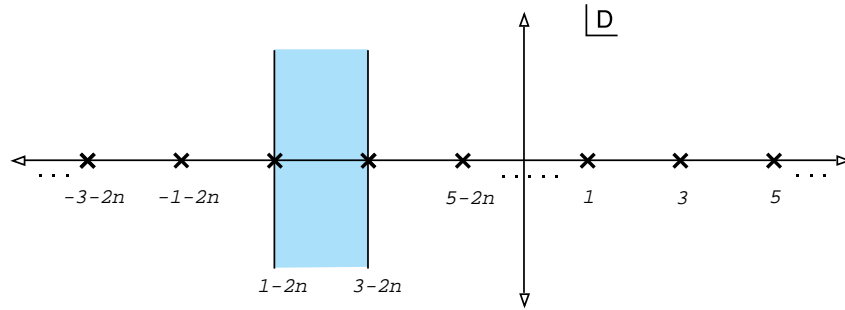


Figure 2. The complex D -plane. The integral $I_n(\mu)$ is given by equation (11) within the shaded region of the D -plane. Poles are represented by crosses.

which is valid for

$$1 - 2n < D < 3 - 2n \tag{12}$$

and $E < 0$. If we are interested in scattering problems in four space–time dimensions we must now perform two analytic continuations: one in the energy E and one in the number of dimensions D . Performing the analytic continuation in E first, by using the evaluation at $E + i\eta$, rather than E itself, we find that for $E > 0$

$$I_n(\mu) = -\frac{M}{4\pi^2} \left(\frac{\mu}{2}\right)^{4-D} (-ME)^n (-ME^+)^{(D-3)/2} \Gamma\left(\frac{2n + D - 1}{2}\right) \Gamma\left(\frac{3 - D - 2n}{2}\right) \tag{13}$$

provided that $1 - 2n < D < 3 - 2n$. The following discussion makes extensive use of this result.

The usual dimensional regularization prescription for calculation of the integral I_n in dimensions where the expression (13) is not valid is to define a new function $\tilde{I}_n(\mu)$ which agrees with the original $I_n(\mu)$ in the region $1 - 2n < D < 3 - 2n$, but is defined by equation (13) in the rest of the complex D -plane. For this \tilde{I}_n we may use the Gamma function identity,

$$\Gamma(n + a)\Gamma(1 - a - n) = \Gamma(a)\Gamma(1 - a)(-1)^n \tag{14}$$

to obtain

$$\tilde{I}_n(\mu) = -\frac{M}{4\pi^2} \left(\frac{\mu}{2}\right)^{4-D} (ME)^n (-ME^+)^{(D-3)/2} \Gamma\left(\frac{D - 1}{2}\right) \Gamma\left(\frac{3 - D}{2}\right) \tag{15}$$

which is correct for all D except odd integers. This expression differs from that of [2, 3] by a factor due to the way the angular integration was performed. Note that if the identity (14) is applied directly to equation (9), then the result of [2, 3] for $\tilde{I}_n(\mu)$ is recovered.

2. Subtraction schemes and the divergent integral

In $D = 4$ equation (15) yields the result of [4]:

$$\tilde{I}_n(\mu) = -p^{2n} \frac{iMp}{4\pi} \tag{16}$$

where $p = \sqrt{ME}$. The regularization scheme corresponding to equation (15) with removal of any poles at the critical dimension is, by convention, minimal subtraction. Of course, there are no poles in the dimension of interest to us, $D = 4$, and so only the imaginary part of the integral survives in minimal subtraction.

The expression (13) has poles at $D = 1 - 2n, -1 - 2n, \dots, D = 3 - 2n, 5 - 2n, 7 - 2n, \dots$. The former lie below the region of analyticity of \tilde{I}_n (see figure 2), and can be considered as infrared divergences that arise when I_n is evaluated in such dimensions. Conversely, the latter poles can be considered ultraviolet divergences of \tilde{I}_n . Some of them lie between the region of analyticity and the critical dimension $D = 4$ which we wish to analytically continue to. In the rest of this paper we follow a suggestion of Kaplan *et al* [2, 3] and consider the effects of cancelling out poles of $\Gamma(\frac{3-D-2n}{2})$ (ultraviolet singularities) and $\Gamma(\frac{2n+D-1}{2})$ (infrared singularities) in the integral \tilde{I}_n . This procedure leads to regularization schemes which differ from the standard dimensional regularization with minimal subtraction result of [4], equation (16).

Of course, after renormalization the on-shell piece of the amplitude T calculated via equation (5) is necessarily the same in any scheme. Otherwise one of the basic tenets of effective field theory, insensitivity to short-distance physics, is violated. So we must ultimately get the same physical result regardless of how many poles we subtract[†]. This equivalence after renormalization is enforced by making appropriate choices for the coefficients C_{2n} . Thus, the behaviour of the coefficients, and hence the power counting for these coefficients, is affected by the scheme we use to define \tilde{I}_n . As we will show in the next section some schemes are more useful for reproducing certain physics in the effective theory.

Consider first the Gamma function poles corresponding to ultraviolet divergences in cut-off regularization. When D is close to $3 - 2m$, with $m \geq 0$, the poles of $\Gamma(\frac{3-D-2n}{2})$ between $D = 3 - 2n$ and $D = 4$ have the structure

$$\frac{M}{4\pi^2} \frac{\mu^{2m+1}}{2^{2m}} p^{2(n-m)} \frac{1}{D - 3 + 2m}. \quad (17)$$

Thus we can cancel these poles out by defining,

$$\tilde{I}_n^{\text{new}} \equiv \tilde{I}_n(\mu) - \frac{M}{2\pi^2} \sum_{m=0}^n \left(\frac{\mu}{2}\right)^{2m+1} p^{2(n-m)} \frac{1}{D - 3 + 2m} \quad (18)$$

where $\tilde{I}_n(\mu)$ is given by expression (13).

Similarly, to cancel the poles of $\Gamma(\frac{3-D-2n}{2})$ in $D > 4$ dimensions we must add a term

$$\frac{M}{2\pi^2} \sum_{m=1}^{\infty} p^{2(n+m)} \left(\frac{\mu}{2}\right)^{1-2m} \frac{1}{2m + 3 - D}. \quad (19)$$

Thus if we wish to cancel all poles of the expression (13) for dimensions $D > 3 - 2n$ we must define

$$\tilde{I}_n^{\text{uvPDS}} \equiv \tilde{I}_n(\mu) - \frac{M}{2\pi^2} \sum_{m=-\infty}^n \left(\frac{\mu}{2}\right)^{2m+1} p^{2(n-m)} \frac{1}{D - 3 + 2m}. \quad (20)$$

This subtracts the poles which correspond to ultraviolet divergences for $D > 3 - 2n$. We refer to the resulting regularization scheme as dimensional regularization with ultraviolet power-law divergence subtraction (uvPDS).

It is now a straightforward matter to show that uvPDS is precisely equivalent to cut-off regularization in $D = 4$. For the pieces of the sum with $m \geq 0$ we find

$$\begin{aligned} & -\frac{M}{2\pi^2} \sum_{m=0}^n \left(\frac{\mu}{2}\right)^{2m+1} p^{2(n-m)} \frac{1}{1 + 2m} \\ &= -\frac{M}{2\pi^2} \left[\left(\frac{\mu}{2}\right) p^{2n} + \left(\frac{\mu}{2}\right)^3 \frac{p^{2n-2}}{3} + \dots + \left(\frac{\mu}{2}\right)^{2n+1} \frac{1}{2n + 1} \right] \end{aligned} \quad (21)$$

[†] This point has been stressed in the context of effective field theories of the nucleon–nucleon interaction by van Kolck [5].

and for the pieces which arise from poles in $D > 4$ it follows from the observation that

$$\frac{M}{2\pi^2} \sum_{m=1}^{\infty} p^{2m} \left(\frac{\mu}{2}\right)^{-2m+1} \frac{1}{2m-1} = \frac{Mp}{4\pi^2} \log\left(\frac{\mu/2+p}{\mu/2-p}\right) \quad (22)$$

provided that p is kept less than $\mu/2$. It is clear that if we identify $\beta = \mu/2$, then in $D = 4$ \tilde{I}_n^{new} in uvPDS is precisely equivalent to the cut-off result of equation (8).

3. Scaling of effective field theory coefficients

The choice of \tilde{I}_n made by Kaplan *et al*, known as power-law divergence subtraction (PDS) is to cancel the pole of expression (15) in $D = 3$ for all n [2, 3]. Of course, as we have seen the PDS scheme is only one of a much more general class of definitions of the integral \tilde{I}_n . Kaplan *et al*'s main motivation in considering the PDS scheme was to produce a consistent power counting in non-relativistic effective theories where there is an unnaturally large scattering length. Much blood, sweat, and ink has been spilt over this problem during the past few years [2–26]. The PDS approach to this difficulty is elegant, and leads to good power counting for the coefficients in the low-energy Lagrangian. Here we show that in fact PDS is the simplest scheme of this type which leads to good power-counting for the bare coefficients C_{2n} . It has been shown how to obtain similar power-counting for the on-shell T-matrix within the framework of any regularization scheme [5, 18]. However, [5, 18] do not address the issue of the scaling of the coefficients in the bare EFT Lagrangian. The power counting of [2, 3, 5, 18] has also been derived within the framework of the Wilsonian renormalization group [21].

3.1. Review of coefficient scaling with the PDS definition of the divergent integral

The PDS choice is to retain only the $m = 0$ term of the sum (20). This corresponds to the linear divergence in the cut-off approach. The integral \tilde{I}_n is defined to be (throughout this section we work in $D = 4$):

$$\tilde{I}_n^{\text{PDS}} = -p^{2n} \frac{M}{4\pi} \left(ip + \frac{\mu}{\pi} \right) \quad (23)$$

where our result for the second term differs by a factor of π from that of [2, 3] because the angular integration was performed in $D = 4$ rather than $D = 3$.

This definition of \tilde{I}_n leads to a straightforward solution of the Lippmann–Schwinger equation, (5):

$$\frac{1}{T^{\text{on}}(p)} = \frac{1}{\sum_{n=0}^{\infty} C_{2n} p^{2n}} + \frac{M\mu}{4\pi^2} + \frac{iMp}{4\pi}. \quad (24)$$

Now, we attempt to match this to a form of the inverse amplitude which corresponds to the presence of an unnaturally large scattering length

$$\frac{1}{T^{\text{on}}(p)} = -\frac{M}{4\pi} \left(-\frac{1}{a} + \frac{1}{2} r_e p^2 + \mathcal{O}(p^4) - ip \right) \quad (25)$$

with $1/a \ll \Lambda_0$, and $r_e \sim 1/\Lambda_0$, where Λ_0 is the ‘natural’ scale set by the theory underlying the low-energy effective theory. Matching expressions (24) and (25) for $\Lambda_0 > p > 1/a$ it is straightforward to deduce [2, 3] that the coefficients scale as

$$C_0 \sim \frac{1}{M\mu} \quad C_{2n} \sim \frac{1}{M\mu^{n+1}\Lambda_0^n} \quad (26)$$

provided that the scale μ is kept greater than, or of the order of, the scale $1/a$. As pointed out by Kaplan *et al* we may now choose the scale μ to be of order p .

With this choice all loops involving the operator C_0 are of the same order and so the operator $C_0(\psi^\dagger\psi)^2$ must be treated non-perturbatively. This power counting can be understood from the viewpoint of the renormalization group [3, 21] as an expansion around a non-trivial IR fixed point. This fixed point describes systems with a bound state at exactly zero energy. In the PDS scheme, the potential at the fixed point is independent of energy or momentum and scales like $1/\mu$, as in equation (26).

From equation (26) we see that the effects of the operators $C_{2n}\hat{p}^{2n}$ are suppressed by a factor $(p/\Lambda_0)^n$ relative to the non-perturbative effects of the dimension-six operator. Therefore, these higher-dimensional operators may be treated perturbatively. Hence we can re-expand the first term in equation (24) in the form

$$\frac{1}{\sum_{n=0}^{\infty} C_{2n} p^{2n}} = \frac{1}{C_0} - \frac{C_2}{C_0^2} p^2 + \left[\left(\frac{C_2}{C_0^2} \right)^2 - \frac{C_4}{C_0^2} \right] p^4 + \dots \quad (27)$$

The effective range expansion (25) can then be reproduced to any desired order in this p/Λ_0 expansion.

In an RG approach the coefficients in the effective range expansion are in one-to-one correspondence with the coefficients of the RG eigenfunctions in the expansion of the potential around the fixed point. These coefficients scale with definite powers of the renormalization scale μ in the PDS scheme or the cut-off in a Wilsonian approach [21]. Although the scaling behaviour is the same in the two approaches, the detailed form of the potential is not: the fixed-point and RG eigenfunctions all have a more complicated energy dependence when a cut-off is used.

3.2. Scaling in PDS in a natural theory

Note that in a natural theory, where $1/a \sim \Lambda_0$, matching (25) to (24) in fact yields

$$C_{2n} \sim \frac{1}{M \Lambda_0^{2n+1}} \quad (28)$$

provided only that $\mu < \Lambda_0$. Since the scale μ does not enter these scaling relations we are free to choose $\mu = 0$, which is, of course, the choice of minimal subtraction. However $\mu \neq 0$ with $p \sim \mu$ is also a choice that leads to good power counting. Clearly the tree-level effects of the operators C_{2n} are suppressed by a power $(p/\Lambda_0)^{2n}$. Moreover, the effect of a loop with superficial degree of divergence $2n + 1$ is down by a factor of $(p/\Lambda_0)^{2n+1}$, relative to the tree-level C_0 . Thus in the natural theory there is no need to treat the effects of C_0 non-perturbatively, and the effects of higher-derivative operators are more strongly suppressed than in the unnatural case.

3.3. Scaling in uvPDS

We can now deduce the effect of modifying the definition of \tilde{I}_n by adding the terms for $m = 1$ to $m = n$ which appear in equation (20). Recall that if a cut-off regulator is used these terms correspond to cubic and higher power-law divergences. Attempting to solve the theory exactly, as was done above for the PDS subtraction, becomes much more complicated (see [15, 16, 22] for examples), as each integral \tilde{I}_n has a different number of subtractions. This complexity gives rise to nonlinear relations between the bare and renormalized coefficients. However, the crucial point in any such attempt is that if we keep the scale μ well below the scale Λ_0 then the effect of all power-law divergences will be suppressed by powers of μ/Λ_0 , and so they make no change to the scaling of coefficients quoted above. For instance, if we examine a loop as

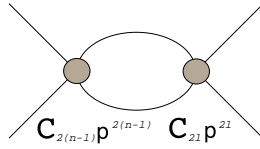


Figure 3. Characteristic loop graph.

shown in figure 3 with one insertion of $C_{2(n-l)}$ and one of C_{2l} then, considering for the present only the pieces of the sum in equation (20) corresponding to power-law divergences of degree three and above, the loop becomes

$$C_{2(n-l)}C_{2l} \frac{M}{2\pi^2} \sum_{m=1}^n \left(\frac{\mu}{2}\right)^{2m+1} p^{2(n-m)} \frac{1}{1+2m}. \tag{29}$$

Given the scaling above for the coefficients C_{2j} the m th term here produces scaling of the coefficient $C_{2(n-m)}$ of the form

$$C_{2(n-m)} \sim \frac{1}{M\mu^{n-2m+1}\Lambda_0^n}. \tag{30}$$

Compared with the leading scaling of this coefficient $C_{2(n-m)} \sim 1/(M\mu^{n-m+1}\Lambda_0^{n-m})$ the scaling (30) is suppressed by a factor of $(\mu/\Lambda_0)^m$. Thus, even if the terms in equation (20) which correspond to power-law divergences of higher degree than those considered in PDS were included they would not modify the PDS scaling (26).

In contrast, the terms arising from poles in $D = 5$ and above, which sum up to produce the logarithm of equation (22), *would* modify the PDS scaling (26) if they were included in the definition of \tilde{I}_n . To see this let us only retain the terms of the sum in equation (20) for $m \leq 0$. Then

$$\tilde{I}_n = -p^{2n} \frac{M}{4\pi} \left(ip + \frac{\mu}{\pi} - \frac{1}{\pi} \log \left(\frac{\mu + 2p}{\mu - 2p} \right) \right) \tag{31}$$

so solving the Lippmann–Schwinger equation gives

$$\frac{1}{T^{\text{on}}(p)} = \frac{1}{\sum_{n=0}^{\infty} C_{2n} p^{2n}} + \frac{M\mu}{4\pi^2} - \frac{Mp}{4\pi^2} \log \left(\frac{\mu + 2p}{\mu - 2p} \right) + \frac{iMp}{4\pi}. \tag{32}$$

Expanding out the log in powers of p/μ it is straightforward to see that matching to the expression (25) requires

$$C_{2n} = \frac{1}{M\mu^{2n+1}}. \tag{33}$$

This, of course, is the same scaling one would expect in a cut-off theory in which the cut-off was kept below the scale Λ_0 [11, 17].

Now, if the coefficients do scale in this way then for momenta $p \sim \mu$ the effect of the higher-derivative operators $C_{2n} p^{2n}$ is *not* suppressed relative to that of the ‘lowest-order’ operator C_0 . Thus if uvPDS is used to regulate the divergent loop integral the resulting power counting for the C is such that one cannot justify any truncation of the sum over all higher-derivative operators. Consequently, we conclude that the choice (20) does not lead to good power counting for the coefficients C_{2n} .

The physical reason this occurs is that the inclusion of terms that contain negative powers of μ in the definition of the integral $\tilde{I}_n^{\text{uvPDS}}$ introduces energy dependence in the amplitude at scale μ . Since μ is much less than the natural scale of energy dependence Λ_0 , this energy dependence is unnaturally rapid. It must be cancelled by using larger values of the coefficients C_{2n} . In fact as shown in [21], this occurs automatically within an RG treatment where the

unnatural energy dependence is absorbed into the forms of the fixed-point potential and the RG eigenfunctions for the expansion around it. In our case we would have the fixed-point potential

$$V(p) = \left(-\frac{M\mu}{4\pi^2} + \frac{Mp}{4\pi^2} \log \left(\frac{\mu + 2p}{\mu - 2p} \right) \right)^{-1}. \quad (34)$$

If this potential is iterated via the Lippmann–Schwinger equation and uvPDS (which we have already shown is equivalent to a cut-off) applied to define the divergent integral then all the spurious energy dependence introduced by the terms in the sum for \tilde{I}_n with negative powers of μ are cancelled. The analysis given here shows that this is essentially equivalent to just ignoring the effects of the poles in $D > 5$ which lead to logarithmic behaviour in the inverse amplitude altogether.

One difference between uvPDS (or a cut-off) and PDS *à la* KSW is that it distinguishes between energy and momentum dependence in the potential. The RG eigenfunctions that correspond to the terms in the effective range expansion are purely energy dependent [21]. Ultimately they can be thought of as terms in an energy-dependent pseudopotential that acts as an energy-dependent boundary condition on the wavefunction at the origin [5, 21]. There are in addition momentum-dependent eigenfunctions with different scaling behaviours, but these do not contribute to the on-shell scattering amplitude.

4. Summary

The integral \tilde{I}_n is a key ingredient of non-relativistic effective field theories in general, and EFTs of the nucleon–nucleon interaction in particular. Since this integral has infinitely many poles in the complex space of dimensions there are infinitely many ways to define it, depending on how many of these poles we choose to subtract in our redefined \tilde{I}_n . The original suggestion of [2] which removes the pole at $D = 3$ therefore exists amidst myriad alternative schemes. The alternative corresponding to removing *all* of the poles which arise due to ultraviolet divergences of \tilde{I}_n exactly reproduces the result obtained by simply regulating I_n via a cut-off.

We have also analysed the way that coefficients in the effective field theory Lagrangian scale in regularization schemes using these more general definitions of \tilde{I}_n . This analysis shows that, in the presence of an unnaturally large scattering length, PDS is the simplest definition of a subtracted \tilde{I}_n which leads to good power counting for the coefficients that appear in the effective field theory Lagrangian.

Acknowledgments

DRP acknowledges useful discussions on the nature of dimensional regularization with I R Afnan, T D Cohen, and A Schreiber. DRP and SRB thank the US Department of Energy, Nuclear Physics Division, for its support (grant DE-FG02-93ER-40762). MCB is grateful for useful comments from J A McGovern, and for support from the UK EPSRC.

Appendix A. ‘Maximal’ subtraction

Here we consider the role played by the poles in $D < 1 - 2n$. As mentioned above, these correspond to infrared singularities of the original integrand. Since the integral of interest has no infrared divergences in $D = 4$, this discussion is not particularly relevant to nature. However, it leads to an intriguing result, and so we present it in this appendix.

To cancel the poles of $\Gamma(\frac{2n+D-1}{2})$ in $D < 1 - 2n$ dimensions we must add a term

$$-\frac{M}{2\pi^2} \sum_{m=n+1}^{\infty} \left(\frac{\mu}{2}\right)^{2m+1} p^{2(n-m)} \frac{1}{D-3+2m}. \quad (35)$$

We can then define a scheme which we call maximal subtraction, that subtracts *all* poles in the complex D -plane:

$$\tilde{I}_n^{\text{all}} \equiv \tilde{I}_n(\mu) - \frac{M}{2\pi^2} \sum_{m=-\infty}^{\infty} \left(\frac{\mu}{2}\right)^{2m+1} p^{2(n-m)} \frac{1}{D-3+2m} \quad (36)$$

in all dimensions D in the complex D -plane. We now specialize to the case $D = 4$.

The sum in equation (36) is divergent for any value of the momentum p . In order to define a sum for this series we first define the function $f(p)$ via

$$f(p) = -\frac{M}{2\pi^2} \sum_{m=0}^{\infty} \left(\frac{\mu}{2}\right)^{2m+1} p^{2(n-m)} \frac{1}{2m+1}. \quad (37)$$

If $\mu/2 < p$ then this series may be summed to yield

$$f(p) = -\frac{Mp}{4\pi^2} p^{2n} \log\left(\frac{p+\mu/2}{p-\mu/2}\right). \quad (38)$$

In the region of the complex p -plane, $\mu/2 < |p| < R$, with R some large real number, this function is an analytic function of p . Hence we can define a function $\tilde{f}(p)$ which is the analytic continuation of $f(p)$ into the region $0 \leq |p| < R$. We make the analytic continuation

$$\tilde{f}(p) = -\frac{Mp}{4\pi^2} p^{2n} \log\left(\left|\frac{p+\mu/2}{p-\mu/2}\right|\right) + p^{2n} \frac{iMp}{4\pi}. \quad (39)$$

We now define $\tilde{I}_n^{\text{MaxS}}$ to be the result found if the sum over positive m in equation (36) is replaced by its analytic continuation \tilde{f} , thus:

$$\tilde{I}_n^{\text{MaxS}} = \tilde{I}_n(\mu) + p^{2n} \frac{iMp}{4\pi}. \quad (40)$$

Taking the limit $D \rightarrow 4$ and evaluating the (finite) integral $\tilde{I}_n(\mu)$ implies

$$\tilde{I}_n^{\text{MaxS}} = 0. \quad (41)$$

By subtracting all the poles in the complex D -plane we have forced a definition of the value of the divergent integral I_n which gives zero.

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